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Compatible Poisson brackets of hydrodynamic type

E V Ferapontov

Department of Mathematical Sciences, Loughborough University, Loughborough, Leicestershire LE11 3TU, UK
and
Centre for Nonlinear Studies, Landau Institute of Theoretical Physics, Academy of Science of Russia, Kosygina 2, 117940 Moscow, GSP-1, Russia

E-mail: E.V.Ferapontov@lboro.ac.uk and fer@landau.ac.ru

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Abstract
Some general properties of compatible Poisson brackets of hydrodynamic type are discussed, in particular: (a) an invariant differential–geometric criterion of the compatibility based on the Nijenhuis tensor which is slightly different from those existing in the literature; (b) the Lax pair with a spectral parameter governing compatible Poisson brackets in the diagonalizable case; (c) the connection of this problem with the class of surfaces in Euclidean space which possess non-trivial deformations preserving the Weingarten operator.

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1. Introduction

In 1983 Dubrovin and Novikov [3] introduced the Poisson brackets of hydrodynamic type

$$\{F, G\} = \int \frac{\delta F}{\delta u^i} A^{ij} \frac{\delta G}{\delta u^j}\, \mathrm{d}x \tag{1}$$

defined by the Hamiltonian operators A^{ij} of the form

$$A^{ij} = g^{ij}\frac{\mathrm{d}}{\mathrm{d}x} + b^{ij}_k u^k_x \qquad b^{ij}_k = -g^{is}\Gamma^j_{sk}. \tag{2}$$

They proved that in the non-degenerate case (det $g^{ij} \neq 0$) the bracket (1) and (2) is skew-symmetric and satisfies the Jacobi identities if and only if the metric g^{ij} (with upper indices) is flat, and Γ^j_{sk} are the Christoffel symbols of the corresponding Levi-Civita connection.

Let us assume that there is a second Poisson bracket of hydrodynamic type defined on the same phase space by the Hamiltonian operator,

$$\tilde{A}^{ij} = \tilde{g}^{ij}\frac{\mathrm{d}}{\mathrm{d}x} + \tilde{b}^{ij}_k u^k_x \qquad \tilde{b}^{ij}_k = -\tilde{g}^{is}\tilde{\Gamma}^j_{sk} \tag{3}$$

corresponding to a flat metric $\tilde{g}^{ij}$. Two Poisson brackets (Hamiltonian operators) are called compatible if their linear combinations $\tilde{A}^{ij} + \lambda A^{ij}$ are Hamiltonian as well. This requirement implies, in particular, that the metric $\tilde{g}^{ij} + \lambda g^{ij}$ must be flat for any λ (plus certain additional restrictions). The necessary and sufficient conditions of the compatibility were first formulated by Dubrovin [6, 7] (see Mokhov [19, 20] for further discussions). In section 2 we reformulate these conditions in terms of the operator $r^i_j = \tilde{g}^{is} g_{sj}$ (theorem 1) which, in particular, implies the vanishing of the Nijenhuis tensor of the operator r^i_j:

$$N^i_{jk} = r^s_j \partial_s r^i_k - r^s_k \partial_s r^i_j - r^i_s(\partial_j r^s_k - \partial_k r^s_j) = 0$$

(see [8, 20]).

Examples of compatible Hamiltonian pairs naturally arise in the theory of Hamiltonian systems of hydrodynamic type (see, e.g., [1, 15, 16, 22, 23, 26]). Dubrovin developed a deep theory for a particular class of compatible Poisson brackets arising within the framework of the associativity equations [6, 7]. Compatible Poisson brackets of hydrodynamic type can also be obtained as a result of Whitham averaging (dispersionless limit) from the local compatible Poisson brackets of integrable systems [3–5, 11, 25, 27, 29]. Some further examples and partial classification results can be found in [8, 11, 14, 19, 22, 24].

If the spectrum of r^i_j is simple, the vanishing of the Nijenhuis tensor implies the existence of a coordinate system where both metrics g^{ij} and $\tilde{g}^{ij}$ become diagonal. In these diagonal coordinates the compatibility conditions take the form of an integrable reduction of the Lamé equations. We present the corresponding Lax pairs in section 3. Another approach to the integrability of this system has been proposed recently by Mokhov [21] by an appropriate modification of Zakharov's scheme [30].

The main observation of this paper is the relationship between compatible Poisson brackets of hydrodynamic type and hypersurfaces $M^{n-1} \in E^n$ which possess non-trivial deformations preserving the Weingarten operator. For surfaces $M^2 \in E^3$ these deformations have been investigated by Finikov and Gambier as long ago as 1933 [12, 13] (see also [2]). In section 4 we demonstrate that the n-orthogonal coordinate system in E^n corresponding to the flat metric $\tilde{g}^{ij} + \lambda g^{ij}$ (rewritten in the diagonal coordinates) deforms with respect to λ in such a way that the Weingarten operators of the coordinate hypersurfaces are preserved up to constant scaling factors. In section 5 we discuss surfaces $M^2 \in E^3$ which possess non-trivial one-parameter deformations preserving the Weingarten operator and explicitly introduce a spectral parameter in the corresponding Gauss–Codazzi equations.

2. Differential–geometric criterion of compatibility

To formulate the necessary and sufficient conditions of compatibility we introduce the operator $r^i_j = \tilde{g}^{is} g_{sj}$, which is automatically symmetric

$$r^i_s g^{sj} = r^j_s g^{si} \tag{4}$$

so that $\tilde{g}^{ij} = r^i_s g^{sj} = r^j_s g^{si} = r^{ij}$. In what follows we use the first metric g^{ij} for raising and lowering the indices.

Theorem 1. *Hamiltonian operators (2) and (3) are compatible if and only if the following conditions are satisfied*

(a) The Nijenhuis tensor of r^i_j vanishes:

$$N^i_{jk} = r^s_j \partial_s r^i_k - r^s_k \partial_s r^i_j - r^i_s(\partial_j r^s_k - \partial_k r^s_j) = 0. \tag{5}$$

(b) The metric coefficients $\tilde{g}^{ij} = r^{ij}$ *satisfy the equations*

$$\nabla^i \nabla^j r^{kl} + \nabla^k \nabla^l r^{ij} = \nabla^i \nabla^k r^{jl} + \nabla^j \nabla^l r^{ik}. \tag{6}$$

Here $\nabla^i = g^{is}\nabla_s$ *is the covariant differentiation corresponding to the metric* g^{ij}. *The vanishing of the Nijenhuis tensor implies the following expression for the coefficients* $\tilde{b}^{ij}_k$ *in terms of* r^i_j*:*

$$2\tilde{b}^{ij}_k = \nabla^i r^j_k - \nabla^j r^i_k + \nabla_k r^{ij} + 2b^{sj}_k r^i_s. \tag{7}$$

In a somewhat different form the necessary and sufficient conditions of the compatibility were formulated in [6, 7, 19, 20].

Remark. The criterion of the compatibility of the Hamiltonian operators of hydrodynamic type resembles that of the finite-dimensional Poisson bivectors: two skew-symmetric Poisson bivectors ω^{ij} and $\tilde{\omega}^{ij}$ are compatible if and only if the Nijenhuis tensor of the corresponding recursion operator $r^i_j = \tilde{\omega}^{is}\omega_{sj}$ vanishes. We emphasize that in our situation the operator r^i_j does not coincide with the recursion operator.

Proof of theorem 1. Recall that in terms of g^{ij} and b^{ij}_k the conditions for the operator A to be Hamiltonian take the form

$$2b^{ki}_s g^{sj} = g^{js}\partial_s g^{ik} + g^{ks}\partial_s g^{ij} - g^{is}\partial_s g^{kj} \tag{8}$$

and

$$g^{js}\partial_s b^{ik}_n - g^{is}\partial_s b^{jk}_n + (b^{ij}_s - b^{ji}_s)b^{sk}_n + b^{ik}_s b^{js}_n - b^{jk}_s b^{is}_n = 0 \tag{9}$$

respectively (the last condition follows from the identity $g^{il}g^{js}R^k_{nls} = 0$ after rewriting it in terms of b^{ij}_k). Note that (8) is equivalent to a pair of simpler conditions

$$b^{ij}_k + b^{ji}_k = \partial_k g^{ij} \qquad b^{ik}_s g^{sj} = b^{jk}_s g^{si}.$$

To write down the compatibility conditions of (2) and (3), we replace g^{ij} and b^{ij}_k by the linear combinations

$$g^{ij} \to \lambda g^{ij} + \tilde{g}^{ij} \qquad b^{ij}_k \to \lambda b^{ij}_k + \tilde{b}^{ij}_k$$

substitute them into (8) and (9), collect the terms with λ (terms with λ^2 and λ^0 vanish since (2) and (3) are Hamiltonian) and equate them to zero. Thus, equation (8) produces the first compatibility condition

$$2\tilde{b}^{ki}_s g^{sj} + 2b^{ki}_s \tilde{g}^{sj} = \tilde{g}^{js}\partial_s g^{ik} + g^{js}\partial_s \tilde{g}^{ik} + \tilde{g}^{ks}\partial_s g^{ij} + g^{ks}\partial_s \tilde{g}^{ij} - \tilde{g}^{is}\partial_s g^{kj} - g^{is}\partial_s \tilde{g}^{kj}. \tag{10}$$

Similarly, equation (9) produces the second compatibility condition

$$\tilde{g}^{js}\partial_s b^{ik}_n + g^{js}\partial_s \tilde{b}^{ik}_n - \tilde{g}^{is}\partial_s b^{jk}_n - g^{is}\partial_s \tilde{b}^{jk}_n + (\tilde{b}^{ij}_s - \tilde{b}^{ji}_s)b^{sk}_n + (b^{ij}_s - b^{ji}_s)\tilde{b}^{sk}_n + \tilde{b}^{ik}_s b^{js}_n + b^{ik}_s \tilde{b}^{js}_n$$
$$-\tilde{b}^{jk}_s b^{is}_n - b^{jk}_s \tilde{b}^{is}_n = 0. \tag{11}$$

To simplify further calculations, it is convenient to work in the coordinates where the flat metric g assumes the constant coefficient form $g^{ij} = \text{constant}$, so that $b^{ij}_k \equiv 0$. In these coordinates the compatibility conditions (10) and (11) reduce to

$$2\tilde{b}^{ki}_s g^{sj} = g^{js}\partial_s \tilde{g}^{ik} + g^{ks}\partial_s \tilde{g}^{ij} - g^{is}\partial_s \tilde{g}^{kj} \tag{12}$$

and

$$g^{js}\partial_s\tilde{b}^{ik}_n - g^{is}\partial_s\tilde{b}^{jk}_n = 0 \tag{13}$$

respectively. Rewriting the left-hand side of (12) in the form $2\tilde{b}^{ki}_s g^{sj} = 2\tilde{b}^{ki}_s\tilde{g}^{sl}(r^{-1})^j_l$ and substituting the expressions for $\tilde{b}^{ki}_s\tilde{g}^{sl}$ from (8), we arrive at

$$(\underline{\tilde{g}^{ls}\partial_s\tilde{g}^{ik}} + \tilde{g}^{ks}\partial_s\tilde{g}^{il} - \tilde{g}^{is}\partial_s\tilde{g}^{kl})(r^{-1})^j_l = \underline{g^{js}\partial_s\tilde{g}^{ik}} + g^{ks}\partial_s\tilde{g}^{ij} - g^{is}\partial_s\tilde{g}^{kj}.$$

Cancelling the underlined terms and substituting $\tilde{g}^{ij} = r^i_s g^{sj} = r^j_s g^{si}$, we obtain

$$(r^s_p g^{pk}\partial_s(r^l_n g^{ni}) - r^s_p g^{pi}\partial_s(r^l_n g^{nk}))(r^{-1})^j_l = g^{ks}\partial_s(r^j_n g^{ni}) - g^{is}\partial_s(r^j_n g^{nk})$$

or

$$(r^s_p g^{pk} g^{ni}\partial_s r^l_n - r^s_p g^{pi} g^{nk}\partial_s r^l_n)(r^{-1})^j_l = g^{ks} g^{ni}\partial_s r^j_n - g^{is} g^{nk}\partial_s r^j_n.$$

Contraction with r^m_j results in

$$g^{pk} g^{ni} r^s_p\partial_s r^m_n - g^{pi} g^{nk} r^s_p\partial_s r^m_n = g^{ks} g^{ni} r^m_j\partial_s r^j_n - g^{is} g^{nk} r^m_j\partial_s r^j_n$$

which is equivalent to

$$g^{pk} g^{ni}(r^s_p\partial_s r^m_n - r^s_n\partial_s r^m_p - r^m_j\partial_p r^j_n + r^m_j\partial_n r^j_p) = 0$$

implying the vanishing of the Nijenhuis tensor.

To establish the second identity (9), we will make use of the formula (7) for the coefficients $\tilde{b}^{ij}_k$ in terms of r, the proof of which is included in the appendix (note that this formula is true in an arbitrary coordinate system). In the coordinates where $g^{ij} =$ constant we have $g^{is}\partial_s = \nabla^i$, $2\tilde{b}^{ij}_k = \nabla^i r^j_k - \nabla^j r^i_k + \nabla_k r^{ij}$, so that (13) takes the form

$$\nabla^j(\underline{\nabla^i r^k_n} - \nabla^k r^i_n + \nabla_n r^{ik}) - \nabla^i(\underline{\nabla^j r^k_n} - \nabla^k r^j_n + \nabla_n r^{jk}) = 0.$$

Cancellation of the underlined terms and contraction with g^{sn} produces (6). This completes the proof of the theorem. □

Remark. If the spectrum of r^i_j is simple, condition (6) is redundant: it is automatically satisfied by virtue of (5) and the flatness of both metrics g and $\tilde{g}$ (indeed, in this case equations (16) and (17) of section 3 already imply the compatibility). This was the motivation for me to drop condition (6) in the compatibility criterion formulated in [9]. However, in this general form the criterion proved to be incorrect: recently it was pointed out by Mokhov [20] that in the case when the spectrum of r^i_j is not simple the vanishing of the Nijenhuis tensor is no longer sufficient for compatibility.

3. Compatibility conditions in the diagonal form: the Lax pairs

If the spectrum of r^i_j is simple, the vanishing of the Nijenhuis tensor implies the existence of the coordinates $R^1, \ldots, R^n$ in which the objects $r^i_j, g^{ij}, \tilde{g}^{ij}$ become diagonal. Moreover, the ith eigenvalue of r^i_j depends only on the coordinate R^i, so that $r^i_j = \mathrm{diag}(\eta_i)$, $g^{ij} = \mathrm{diag}(g^{ii})$, $\tilde{g}^{ij} = \mathrm{diag}(g^{ii}\eta_i)$, where η_i is a function of R^i. This is a generalization of the analogous observation by Dubrovin [6] in the particular case of compatible Poisson brackets originating from the theory of the associativity equations. Introducing the Lamé coefficients H_i and the rotation coefficients β_{ij} by the formulae

$$H_i = \sqrt{g_{ii}} = 1/\sqrt{g^{ii}} \qquad \partial_i H_j = \beta_{ij} H_i \tag{14}$$

we can rewrite the zero-curvature conditions for the metric g in the form

$$\partial_k \beta_{ij} = \beta_{ik}\beta_{kj} \tag{15}$$

$$\partial_i \beta_{ij} + \partial_j \beta_{ji} + \sum_{k\neq i,j} \beta_{ki}\beta_{kj} = 0. \tag{16}$$

The zero-curvature condition for the metric $\tilde{g}$ imposes the additional constraint

$$\eta_i \partial_i \beta_{ij} + \eta_j \partial_j \beta_{ji} + \tfrac{1}{2}\eta_i' \beta_{ij} + \tfrac{1}{2}\eta_j' \beta_{ji} + \sum_{k\neq i,j} \eta_k \beta_{ki}\beta_{kj} = 0 \tag{17}$$

resulting from (16) after the substitution of the rotation coefficients $\tilde{\beta}_{ij} = \beta_{ij}\sqrt{\eta_i/\eta_j}$ of the metric $\tilde{g}$. As can be readily seen, equations (16) and (17) already imply the compatibility, so that in the diagonalizable case condition (6) of theorem 1 is indeed superfluous. Solving equations (16) and (17) for $\partial_i \beta_{ij}$, we can rewrite (15)–(17) in the form

$$\begin{aligned} \partial_k \beta_{ij} &= \beta_{ik}\beta_{kj} \\ \partial_i \beta_{ij} &= \frac{1}{2}\frac{\eta_i'}{\eta_j - \eta_i}\beta_{ij} + \frac{1}{2}\frac{\eta_j'}{\eta_j - \eta_i}\beta_{ji} + \sum_{k\neq i,j} \frac{\eta_k - \eta_j}{\eta_j - \eta_i}\beta_{ki}\beta_{kj}. \end{aligned} \tag{18}$$

It can be verified by a straightforward calculation that system (18) is compatible for any choice of the functions $\eta_i(R^i)$, and its general solution depends on $n(n-1)$ arbitrary functions of one variable (indeed, one can arbitrarily prescribe the value of β_{ij} on the jth coordinate line). Under the additional 'Egorov' assumption $\beta_{ij} = \beta_{ji}$, system (18) reduces to that studied by Dubrovin in [7]. For $n \geqslant 3$ system (18) is essentially nonlinear. Its integrability follows from the Lax pair

$$\partial_j \psi_i = \beta_{ij}\psi_j \qquad \partial_i \psi_i = -\frac{\eta_i'}{2(\lambda + \eta_i)}\psi_i - \sum_{k\neq i} \frac{\lambda + \eta_k}{\lambda + \eta_i}\beta_{ki}\psi_k \tag{19}$$

with a spectral parameter λ (another demonstration of the integrability of system (18) has been proposed recently in [21] by an appropriate modification of Zakharov's approach [30]).

Remark. In fact, the Lax pair (19) is gauge-equivalent to the equations

$$(\tilde{g}^{ik} + \lambda g^{ik})\partial_k \partial_j \psi + (\tilde{b}^{ik}_j + \lambda b^{ik}_j)\partial_k \psi = 0$$

for the Casimirs $\int \psi \, \mathrm{d}x$ of the Hamiltonian operator $\tilde{A}^{ij} + \lambda A^{ij}$.

After the gauge transformation $\psi_i = \varphi_i/\sqrt{\lambda + \eta_i}$ the Lax pair (19) assumes the manifestly skew-symmetric form

$$\partial_j \varphi_i = \sqrt{\frac{\lambda + \eta_i}{\lambda + \eta_j}}\beta_{ij}\varphi_j \qquad \partial_i \varphi_i = -\sum_{k\neq i} \sqrt{\frac{\lambda + \eta_k}{\lambda + \eta_i}}\beta_{ki}\varphi_k \tag{20}$$

which is of the type discussed in [17]. Thus, we can introduce an orthonormal frame $\vec{\varphi}_1, \ldots, \vec{\varphi}_n$ in the Euclidean space E^n satisfying the equations

$$\partial_j \vec{\varphi}_i = \sqrt{\frac{\lambda + \eta_i}{\lambda + \eta_j}}\beta_{ij}\vec{\varphi}_j \qquad \partial_i \vec{\varphi}_i = -\sum_{k\neq i} \sqrt{\frac{\lambda + \eta_k}{\lambda + \eta_i}}\beta_{ki}\vec{\varphi}_k \qquad (\vec{\varphi}_i, \vec{\varphi}_j) = \delta_{ij}. \tag{21}$$

Let us introduce a vector $\vec{r}$ such that

$$\partial_i \vec{r} = \frac{H_i}{\sqrt{\lambda + \eta_i}} \vec{\varphi}_i$$

(the compatibility of these equations can be readily verified). In view of the formula

$$(\partial_i \vec{r}, \partial_j \vec{r}) = \frac{H_i^2}{\lambda + \eta_i} \delta_{ij}$$

the radius-vector $\vec{r}$ is descriptive of an n-orthogonal coordinate system in E^n corresponding to the flat metric

$$\sum_i \frac{H_i^2}{\lambda + \eta_i} (\mathrm{d}R^i)^2.$$

Geometrically, $\vec{\varphi}_i$ are the unit vectors along the coordinate lines of this n-orthogonal system.

Let us discuss in some more detail the case $\eta_i = \text{constant} = c_i$, in which the system (18) takes the form

$$\begin{aligned} \partial_k \beta_{ij} &= \beta_{ik} \beta_{kj} \\ \partial_i \beta_{ij} &= \sum_{k \neq i,j} \frac{c_k - c_j}{c_j - c_i} \beta_{ki} \beta_{kj}. \end{aligned} \tag{22}$$

One can readily verify that the quantity

$$P_i = \sum_{k \neq i} (c_k - c_i) \beta_{ki}^2$$

is an integral of system (22), namely, $\partial_j P_i = 0$ for any $i \neq j$, so that P_i is a function of R^i. Utilizing the obvious symmetry $R^i \to s_i(R^i)$, $\beta_{ki} \to \beta_{ki}/s_i'(R^i)$ of system (22), we can reduce P_i to ± 1 (if non-zero). Let us consider the simplest non-trivial case $n = 3$, $P_1 = P_2 = 1$, $P_3 = -1$:

$$P_1 = (c_2 - c_1)\beta_{21}^2 + (c_3 - c_1)\beta_{31}^2 = 1$$

$$P_2 = (c_1 - c_2)\beta_{12}^2 + (c_3 - c_2)\beta_{32}^2 = 1$$

$$P_3 = (c_1 - c_3)\beta_{13}^2 + (c_2 - c_3)\beta_{23}^2 = -1.$$

Assuming $c_3 > c_2 > c_1$ and introducing the parametrization

$$\beta_{21} = \sin p/\sqrt{c_2 - c_1} \qquad \beta_{31} = \cos p/\sqrt{c_3 - c_1}$$

$$\beta_{12} = \sinh q/\sqrt{c_2 - c_1} \qquad \beta_{32} = \cosh q/\sqrt{c_3 - c_2}$$

$$\beta_{13} = \sin r/\sqrt{c_3 - c_1} \qquad \beta_{23} = \cos r/\sqrt{c_3 - c_2}$$

we readily rewrite (22) in the form

$$\partial_1 q = \mu_1 \cos p \qquad \partial_1 r = -\mu_1 \sin p$$

$$\partial_2 p = -\mu_2 \cosh q \qquad \partial_2 r = \mu_2 \sinh q$$

$$\partial_3 p = \mu_3 \cos r \qquad \partial_3 q = \mu_3 \sin r$$

where

$$\mu_1 = \sqrt{\frac{c_3 - c_2}{(c_2 - c_1)(c_3 - c_1)}}$$

$$\mu_2 = \sqrt{\frac{c_3 - c_1}{(c_2 - c_1)(c_3 - c_2)}}$$

$$\mu_3 = \sqrt{\frac{c_2 - c_1}{(c_3 - c_1)(c_3 - c_2)}}.$$

After rescaling, this system simplifies to

$$\begin{aligned} \partial_1 q &= \cos p & \partial_1 r &= -\sin p \\ \partial_2 p &= -\cosh q & \partial_2 r &= \sinh q \\ \partial_3 p &= \cos r & \partial_3 q &= \sin r. \end{aligned} \tag{23}$$

Expressing p and r in the form $p = \arccos \partial_1 q$, $r = \arcsin \partial_3 q$, we can rewrite (23) as a triple of pairwise commuting Monge–Ampère equations

$$\partial_1 \partial_2 q = \cosh q \sqrt{1 - \partial_1 q^2}$$

$$\partial_1 \partial_3 q = -\sqrt{1 - \partial_1 q^2}\sqrt{1 - \partial_3 q^2}$$

$$\partial_2 \partial_3 q = \sinh q \sqrt{1 - \partial_3 q^2}.$$

Similar triples of Monge–Ampère equations were obtained in [10] in the classification of quadruples of 3×3 hydrodynamic-type systems which are closed under the Laplace transformations. However, at the moment there is no explanation of this coincidence.

4. Deformations of n-orthogonal coordinate systems inducing rescalings of the Weingarten operators of the coordinate hypersurfaces

We have demonstrated in section 3 that the radius-vector $\vec{r}(R^1, \ldots, R^n)$ of the n-orthogonal coordinate system in E^n corresponding to the flat diagonal metric $\sum_i \frac{H_i^2}{\lambda + \eta_i} (\mathrm{d}R^i)^2$ satisfies the equations

$$\partial_i \vec{r} = \frac{H_i}{\sqrt{\lambda + \eta_i}} \vec{\varphi}_i$$

where the infinitesimal displacements of the orthonormal frame $\vec{\varphi}_i$ are governed by

$$\partial_j \vec{\varphi}_i = \sqrt{\frac{\lambda + \eta_i}{\lambda + \eta_j}} \beta_{ij} \vec{\varphi}_j \qquad \partial_i \vec{\varphi}_i = -\sum_{k \neq i} \sqrt{\frac{\lambda + \eta_k}{\lambda + \eta_i}} \beta_{ki} \vec{\varphi}_k.$$

Since our formulae depend on the spectral parameter, we may speak of the 'deformation' of the n-orthogonal coordinate system with respect to λ. To investigate this deformation in some more detail, we fix a coordinate hypersurface $M^{n-1} \subset E^n$ (say, R^n = constant). Its radius-vector $\vec{r}$ and the unit normal $\vec{\varphi}_n$ satisfy the Weingarten equations

$$\partial_i \vec{\varphi}_n = \frac{\beta_{ni}}{H_i} \sqrt{\lambda + \eta_n}\, \partial_i \vec{r} \qquad i = 1, \ldots, n-1$$

implying that

$$k^i = \frac{\beta_{ni}}{H_i}\sqrt{\lambda + \eta_n}$$

are the principal curvatures of M^{n-1}. Since η_n is a constant on M^{n-1}, our deformation preserves the Weingarten operator of M^{n-1} up to a constant scaling factor $\sqrt{\lambda + \eta_n}$ (we point out that the curvature line parametrization $R^1, \ldots, R^{n-1}$ is preserved by a construction). Thus, compatible Poisson brackets of hydrodynamic type give rise to deformations of n-orthogonal systems in E^n which, up to scaling factors, preserve the Weingarten operators of the coordinate hypersurfaces. If we follow the evolution of a particular coordinate hypersurface M^{n-1}, this scaling factor can be eliminated by a homothetic transformation of the ambient space E^n, so that we arrive at the non-trivial deformation of a hypersurface which preserves the Weingarten operator. However, this scaling factor cannot be eliminated for all coordinate hypersurfaces simultaneously.

5. Surfaces in E^3 which possess non-trivial deformations preserving the Weingarten operator

Interestingly enough, the problem of the classification of surfaces $M^2 \in E^3$ which possess non-trivial deformations preserving the Weingarten operator has been formulated by Finikov and Gambier as long ago as 1933 [12, 13] (see also Cartan [2]). Among other results, they demonstrated that the only surfaces possessing three-parameter families of such deformations are the quadrics, conformal transforms of surfaces of revolution and all other surfaces having the same spherical image of curvature lines (if surfaces have the same spherical image of curvature lines or, equivalently, related by a Combescure transformation, they can be deformed simultaneously).

In this section we discuss surfaces which possess one-parameter families of such deformations. Let $M^2 \in E^3$ be a surface parametrized by coordinates R^1, R^2 of curvature lines. Let

$$G_{11}(\mathrm{d}R^1)^2 + G_{22}(\mathrm{d}R^2)^2 \tag{24}$$

be its third fundamental form (or metric of the Gaussian image, which is automatically of constant curvature 1). Let k^1, k^2 be the radii of principal curvature satisfying the Peterson–Codazzi equations

$$\frac{\partial_2 k^1}{k^2 - k^1} = \partial_2 \ln\sqrt{G_{11}} \qquad \frac{\partial_1 k^2}{k^1 - k^2} = \partial_1 \ln\sqrt{G_{22}}. \tag{25}$$

Suppose there exists a flat metric

$$g_{11}(\mathrm{d}R^1)^2 + g_{22}(\mathrm{d}R^2)^2 \tag{26}$$

such that

$$G_{11} = g_{11}/\eta_1 \qquad G_{22} = g_{22}/\eta_2 \tag{27}$$

where η_1, η_2 are functions of R^1, R^2, respectively. One can readily verify that under these assumptions the metric

$$\tilde{G}_{11}(\mathrm{d}R^1)^2 + \tilde{G}_{22}(\mathrm{d}R^2)^2 = \frac{g_{11}}{\lambda + \eta_1}(\mathrm{d}R^1)^2 + \frac{g_{22}}{\lambda + \eta_2}(\mathrm{d}R^2)^2 \tag{28}$$

has constant curvature 1 for any λ. Since equations (25) are still true if we replace G_{ii} by $\tilde{G}_{ii}$, we arrive at a one-parameter family of surfaces M^2_λ with the third fundamental forms (28) (which depend on λ) and the principal curvatures k^1, k^2 (which are independent of λ). Hence, the Weingarten operators of surfaces M^2_λ coincide. The problem of the classification of surfaces which possess one-parameter families of deformations preserving the Weingarten operator is thus reduced to the classification of metrics (28) which have constant Gaussian curvatures 1 for any λ. Any such metric generates an infinite family of deformable surfaces whose principal curvatures k^1, k^2 satisfy (25). In terms of the Lamé coefficients $H_1 = \sqrt{g_{11}},\ H_2 = \sqrt{g_{22}}$ and the rotation coefficients $\beta_{12} = \partial_1 H_2/H_1,\ \beta_{21} = \partial_2 H_1/H_2$ our problem reduces to the nonlinear system

$$\begin{aligned}
&\partial_1 H_2 = \beta_{12} H_1 \qquad \partial_2 H_1 = \beta_{21} H_2 \\
&\partial_1\beta_{12} + \partial_2\beta_{21} = 0 \\
&\eta_1\partial_1\beta_{12} + \eta_2\partial_2\beta_{21} + \tfrac{1}{2}\eta_1'\beta_{12} + \tfrac{1}{2}\eta_2'\beta_{21} + H_1H_2 = 0
\end{aligned} \tag{29}$$

which possesses the Lax pair

$$\partial_1\psi = \begin{pmatrix} 0 & -\sqrt{\dfrac{\lambda+\eta_2}{\lambda+\eta_1}}\beta_{21} & \dfrac{H_1}{\sqrt{\lambda+\eta_1}} \\ \sqrt{\dfrac{\lambda+\eta_2}{\lambda+\eta_1}}\beta_{21} & 0 & 0 \\ -\dfrac{H_1}{\sqrt{\lambda+\eta_1}} & 0 & 0 \end{pmatrix}\psi$$

$$\partial_2\psi = \begin{pmatrix} 0 & \sqrt{\dfrac{\lambda+\eta_1}{\lambda+\eta_2}}\beta_{12} & 0 \\ -\sqrt{\dfrac{\lambda+\eta_1}{\lambda+\eta_2}}\beta_{12} & 0 & \dfrac{H_2}{\sqrt{\lambda+\eta_2}} \\ 0 & -\dfrac{H_2}{\sqrt{\lambda+\eta_2}} & 0 \end{pmatrix}\psi.$$

Geometrically, this Lax pair governs infinitesimal displacements of the orthonormal frame of the orthogonal coordinate system on the unit sphere S^2, corresponding to the metric (28). In 2×2 matrices it takes the form

$$2\sqrt{\lambda+\eta_1}\,\partial_1\psi = \begin{pmatrix} \mathrm{i}\sqrt{\lambda+\eta_2}\beta_{21} & H_1 \\ -H_1 & -\mathrm{i}\sqrt{\lambda+\eta_2}\beta_{21} \end{pmatrix}\psi$$

$$2\sqrt{\lambda+\eta_2}\,\partial_2\psi = \mathrm{i}\begin{pmatrix} -\sqrt{\lambda+\eta_1}\beta_{12} & H_2 \\ H_2 & \sqrt{\lambda+\eta_1}\beta_{12} \end{pmatrix}\psi.$$

Remark. In [8] we established a one-to-one correspondence between surfaces possessing non-trivial deformations preserving the Weingarten operator and multi-Hamiltonian systems of hydrodynamic type. Indeed, let us introduce the Hamiltonian operator

$$\delta^{ij}g^{ii}\frac{\mathrm{d}}{\mathrm{d}x} + b^{ij}_k R^k_x$$

associated with the flat diagonal metric (26), and the non-local Hamiltonian operator (see [18])

$$\delta^{ij} G^{ii} \frac{\mathrm{d}}{\mathrm{d}x} + \tilde{b}^{ij}_k R^k_x + R^i_x \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} R^j_x$$

associated with the diagonal metric (24) of constant curvature 1 (these operators are compatible by virtue of (27)). According to the results of Tsarev [28, 29], equations (25) imply that the systems of hydrodynamic type

$$R^1_t = k^1(R) R^1_x \qquad R^2_t = k^2(R) R^2_x$$

are automatically bi-Hamiltonian with respect to both Hamiltonian structures. Characteristic velocities of these systems are the radii of principal curvature of the corresponding surfaces.

The results of this section generalize in a straightforward way to multidimensional hypersurfaces $M^{n-1} \in E^n$.

Appendix. Formula for $\tilde{b}^{ij}_k$

To verify formula (7), it suffices to check the identities

$$\tilde{b}^{ij}_k + \tilde{b}^{ji}_k = \partial_k r^{ij} \tag{A1}$$

$$\tilde{b}^{ik}_s r^{sj} = \tilde{b}^{jk}_s r^{si}. \tag{A2}$$

Substituting the expression for the covariant derivative

$$\nabla_k r^{ij} = \partial_k r^{ij} - b^{si}_k r^j_s - b^{sj}_k r^i_s$$

into (7), we readily obtain

$$2\tilde{b}^{ij}_k = (\nabla^i r^j_k - \nabla^j r^i_k + b^{sj}_k r^i_s - b^{si}_k r^j_s) + \partial_k r^{ij}$$

where the expression in brackets is skew-symmetric in i, j. This proves (A1).

To verify (A2), we first rewrite it in the form

$$(\nabla^i r^k_s - \nabla^k r^i_s + \nabla_s r^{ik} + \underline{2b^{lk}_s r^i_l}) r^{sj} = (\nabla^j r^k_s - \nabla^k r^j_s + \nabla_s r^{jk} + \underline{2b^{lk}_s r^j_l}) r^{si}. \tag{A3}$$

Since $b^{lk}_s r^i_l r^{sj} = b^{lk}_s g_{lt} r^{ti} r^{sj} = -\Gamma^k_{ts} r^{ti} r^{sj}$, the underlined terms cancel in view of the symmetry of Γ. Contracting (A3) with $g_{pj} g_{mk} g_{ni}$, we arrive at

$$g_{pj}(\nabla_n r_{ms} - \nabla_m r_{ns}) r^{sj} + r^s_p \nabla_s r_{mn} = g_{ni}(\nabla_p r_{ms} - \nabla_m r_{ps}) r^{si} + r^s_n \nabla_s r_{mp}$$

which, by virtue of the identity $r_{ms} r^{sj} = r^s_m r^j_s$, transforms to

$$g_{pl} r^l_s (\nabla_n r^s_m - \nabla_m r^s_n) + r^s_p \nabla_s r_{mn} = g_{nl} r^l_s (\nabla_p r^s_m - \nabla_m r^s_p) + r^s_n \nabla_s r_{mp}.$$

In view of the identity

$$r^l_s (\nabla_n r^s_m - \nabla_m r^s_n) = r^s_n \nabla_s r^l_m - r^s_m \nabla_s r^l_n$$

manifesting the vanishing of the Nijenhuis tensor (we emphasize that in (5) partial derivatives can be replaced by covariant derivatives with respect to any symmetric affine connection without changing the Nijenhuis tensor), the last equation can be rewritten as follows:

$$g_{pl}(r^s_n \nabla_s r^l_m - r^s_m \nabla_s r^l_n) + r^s_p \nabla_s r_{mn} = g_{nl}(r^s_p \nabla_s r^l_m - r^s_m \nabla_s r^l_p) + r^s_n \nabla_s r_{mp}$$

or

$$r^s_n \nabla_s r_{pm} - r^s_m \nabla_s r_{pn} + r^s_p \nabla_s r_{mn} = r^s_p \nabla_s r_{nm} - r^s_m \nabla_s r_{np} + r^s_n \nabla_s r_{mp}$$

which is obviously an identity. This proves formula (7).

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